Algebraic Number Theory Final Exam

April 30 2015

This exam is of 50 marks. There are 5 questions, each of 10 marks. Some of them have sub-parts. Please read all the questions carefully and do not cheat. Good luck! (50)

1. Minkowski's bound implies that in every ideal class [I] there is an any ideal $J \in [I]$ with

$$N(J) \leqslant \left(\frac{4}{\pi}\right)^s \cdot \frac{n!}{n^n}\sqrt{|\Delta_K|}.$$

where Δ_K is the discriminant and $n = r + 2s = [K : \mathbb{Q}]$. Use it to compute the class number of
a. $\mathbb{Q}(\sqrt{10})$. (5)
b. $\mathbb{Q}(\sqrt{-11})$ (5)

2a. Let $K = \mathbb{Q}(\sqrt{5})$. Compute the Artin map on primes and use that to find congruence conditions which determine the behaviour of primes - namely which primes split, ramify and are inert. (8)

2b. In particular, determine what happens to the prime 19. (2)

3a.. Let $\mathbb{U}_p$ denote the group of units in $\mathbb{Z}_p$, the valuation ring of $\mathbb{Q}_p$. Show that (5)

$$\mathbb{U}_p = \{x \in \mathbb{Q}_p | |x|_p = 1\}$$

3b Show that $\mathbb{Q}_p^* = \mathbb{Q}_p - \{0\}$ satisfies (5)

$$\mathbb{Q}_p^* \simeq \mathbb{Z} \times \mathbb{U}_p$$

4. Prove that $\mathbb{Z}_p$ is compact with respect to the metric topology induced by the p-adic metric. (Hint: One way to do it is to show it is **complete** and **totally bounded**, where totally bounded means that it can be covered by finitely many ϵ-balls for any $\epsilon > 0$.) (10)

5a. Let $A \subset B$ be integral domains with A integrally closed and B integral over A. Prove that if $\mathfrak{P}$ is a non-zero prime ideal of B then $\mathfrak{P} \cap A$ is a non-zero prime ideal of A. (6)

5b. Let A be a field and B an integral domain containing A which is integral over A. Show that B is a field. (4)